Mathematics and Reality

David Valdman
dmvaldman@gmail.com

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Introduction

How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things?

Albert Einstein

This essay is about mathematics and reality, how they relate and differ. I always feel the strength and imagination inherent within mathematics is abandoned by its terse presentation. Few students are ever aware of just how capable math is at handling what we see before our eyes and within our minds. Sure, we can calculate the trajectory of a comet with stunning accuracy, but to consider ourselves, our universe, and all we see before us—reality, and not simply manifestations of it—in a purely abstract mathematical framework is considered beyond the scope of mathematics and an arrogant breach into the realm of philosophy. But reality in disregard of our experiences—a reality stripped to the mathematics we use to understand it—suddenly transforms into a pristine and integrated place. Forces such as electromagnetism become “connections” on “principle bundles.” These are rigorously defined mathematical concepts independent of physical underpinnings. Matter itself, as corporeal as we imagine it, becomes as abstract as the forces governing it. As ethereal as space and time, matter corresponds to “sections” of an “associated vector bundle,” and mass to a particles’ “drag” through a “Higgs field.” These claims are not infallible (likely, they don’t even make sense, though that is not the intent), but they are what the Standard Model of particle physics asserts.

When our most fundamental physical notions such as time, space, mass, light, etc. come under intense scrutiny we begin to realize just how alien this reality appears and how unprepared we are to deal with its apparent paradoxes. The Nobel Prize winning physicist and one of the fathers of quantum mechanics Paul Dirac was famous for his insightful advice, “follow the math.” Indeed, following the math and leaving our physical intuitions behind has become a necessary and revealing trend in physics. Yet there is glaring irony in this method: mathematics bears no relation to reality! Math is a product of pure reason and so is unconnected to anything existing or real. A number of questions are raised. If reality can be decomposed into math, then just what are we experiencing? What is the *we* that is experiencing *it*? Can physics be completely stripped of the physical, leaving purely mathematical constructions? Can it be that everything we do is simply a response to mathematical constructs, or do these constructs (e.g., mass, a probability wave of a particle) have essential physical meaning? What is reality? Is it around us, or within us? What is math: a tool, an art? This essay will not provide direct answers to such questions, but I hope it will inspire those who seek them.

In fear of being boring, I’ve tried introducing this essay with a shock-and-awe campaign, giving a sense of the ambition before we trudge through the details. In so doing, I provide this essay its own nonstandard geometry: beginning with the near end and moving backwards to justify my outlandish claims. Here is the end of this campaign: a series of excerpts attempting to capture large chunks of our complex reality in the concise language of mathematics.

1. The universe—a four-dimensional affine space A^4. The points of A^4 are called…*events*. The parallel displacements of the universe A^4 constitute a vector space $\mathbf{R}^4$.

2. Time—a linear mapping $t: \mathbf{R}^4 \to \mathbf{R}$ from the vector space of parallel displacements of the universe to the real "time axis." The *time interval* from event $a \in A^4$ to the event $b \in A^4$ is the number $t(b - a)$. If $t(b - a) = 0$, then the events a and b are called *simultaneous*.[1]

Imagine a structured particle, that is, a particle located at some point p in a four-dimensional manifold M ("space-time"), and with an internal structure, or set of states ("spin", etc.) labeled by elements of a complex Lie group G (e.g., SU (2)). In practice we cannot observe this internal structure but only the action of G on some complex vector space V. Thus the total space of all states of such a particle is represented by a vector bundle E over M with fibers isomorphic to V.[2]

Force	*Geometric Description*
Gravity	Curvature of tangent bundle of M
Electromagnetism	Curvature of $U(1)$ bundle on M
Weak	Curvature of $SU(2)$ bundle on M
Strong	Curvature of $SU(3)$ bundle on M[3]

These excerpts run through the Galilean view of the universe, to the more abstract Einsteinian universe and the Standard Model, to the beautiful unification the Standard Model of particle physics provides.[4] We will not go into the details of the mathematics they refer. In fact, it is only better if the terms are unfamiliar. I'd rather leave these passages as curious and mystical as a Zen kōan.

[1] Arnold, V. I. *Mathematical Methods of Classical Mechanics*. Springer, New York, 1978, pg 5.
[2] Darling, R.W.R. *Differential Forms and Connections*. Cambridge Press, 1994, pg 223.
[3] Efthimiou, Costas; Greene, Brian. *Geometrical Concepts in Physics*. Not yet published.
[4] The Standard Model is a quantum field theory consistent with quantum mechanics and special relativity. Established in the early 1970s, it unites the two nuclear forces (strong and weak) with electromagnetism as well as the fundamental particles composing matter. However, the Standard Model is not a complete theory since it does not incorporate gravity. Its dependence on 19 initial parameters (e.g., particle masses) is also discomforting.

Prejudice

The discovery of truth is prevented more effectively, not by the false appearance things present and which mislead into error, not directly by weakness of the reasoning powers, but by preconceived opinion, by prejudice.
Arthur Schopenhauer

Prejudice is the foremost opponent of mathematics. Progress in math has repeatedly been marked by new interpretations of once standard concepts or the creation of new methods to generalize them. If we were really objective creatures, math would have come more easily to us, but we are not, and so we are constantly at odds with our own biases, whether by our apprehension of new concepts (imaginary numbers, for example), or our limited sight in what we consider intuitive or obvious (e.g., Euclid's parallel postulate, or Newtonian physics). In this process, mathematics is constantly being redefined, to the point that no one can give it any global definition.

In my progress, I have time and again been awestruck by the applicability of math's methods. Each new realization usually follows from weeks of confusion and frustration, grappling with some obscure abstraction. We are forever cursed to follow this winding path. I remember reading somewhere that the first few lines in a famous juggling book are, "Take a ball in your hand. Drop it on the floor. Get used to it; you'll be doing it many times." And so too we should all get used to failing at math, for failing is an integral part of discovery. The only consolation is that, like juggling, with enough dedication you will fail only a finite number of times before you get it right forever.

Common sense is the collection of prejudices acquired by age eighteen.
Albert Einstein

Appreciation for mathematics is a struggle. In the beginning math is taught with a sense of unquestionability. It is taught strictly, hiding its flexibility and inventiveness. Curiously, it is only with utmost rigor that the power inherent in math becomes revealed. Many are unsure what is possible and what is forbidden by mathematics because the field seems disconnected with specialized methods for specific circumstances. Rigor is the stripping away of all assumptions, till all that is left are bare essentials; anything else can be tampered with. It is an act of unification: it sifts through assumptions to reveal a common core of things. For instance, most high-school students can easily solve $ax^2 + bx + c = 0$ for x. Yet given $ax^4 + bx^2 + c = 0$, they are at a wall. Learning that one can write the latter equation as $ay^2 + by + c = 0$, where $y = x^2$ (making $x = \pm\sqrt{y}$) shows that there is no additional knowledge needed to solve the problem beyond a deeper realization of the essence of a variable, malleability. x^2 is a variable, just like x. In fact, the *only* difference is that x^2 takes no negative values (over a real domain); otherwise, they cannot be told apart. Who is to say $x \neq z^2$, anyway?

The point of this introduction, and the essay as a whole, is to show that math is not about numbers; it is not about equations. *Mathematics is imagination restricted to consistency*. Numbers and equations are a necessary evil. Their purpose is to give a tangible encapsulation of an abstract idea, one that can be held, written, manipulated. There need be a time when we draw our heads from the clouds and write something on a piece of paper. Even the number 1 is a pure abstraction. If the number 1 actually existed, it would be on exhibit in a museum with crowds of mathematicians waiting to behold its majesty.[5] The benefit of numbers and equations is that they provide an isomorphism, or exact correspondence, to the platonic concepts they define. It is usually best to think what is behind the curtain of an equation.

The concept of graphing a function is a perfect example of this—high-school education gone horribly wrong. What does it mean to say $y = f(x)$? This is a mapping of x as it ranges over the real line, onto the real line ($\mathbf{R} \rightarrow \mathbf{R}$). You plug numbers in, and get numbers out. It is the literal taking of a line, a one-dimensional space—a cut rubber band, if you will—and morphing it appropriately. For instance, let's "do" $y = x^2$. Taking your cut rubber band, fold it in half (since negative numbers become positive), and stretch its end. A more complicated example: $y = \sin(x)$ can similarly be described as coiling a taut string around your hand and elbow. Notice that I'm asking you to *do* something. A function is a verb: a performance on a space. However, when someone asks you what *is* $y = x^2$, you might say it is a parabola: a *thing*. Where is the parabola in this description? The parabola only exists when we consider the function $y = x^2$ as a collection of points (x, x^2). A parabola is a noun: it is the "trace" or the "range" of points (x, x^2) that are the result of the action of mapping $x \rightarrow (x, x^2)$ over some domain. The idea of graphing is only a visualization tool. A mapping, however, is a distortion of a space itself, either by stretching (1-dimensional), bending (2-dimensional), twisting (3-dimensional), or tearing (a discontinuous transformation) its domain. A graph doesn't elucidate this. A graph is not math, it is a mnemonic. However, a graph is efficient at visualizing derivatives, roots, etc. and drawing parallels to orthogonality and dimension. Realize that in this description the x- and y-axes are the *same* thing: the real line, $\mathbf{R}$. The y-axis simply represents the morphed x-axis ($y = f(x\text{-axis})$), and there is no reason for it to be drawn perpendicular except when considering it as a separate and independent dimension explicitly represented by the comma in $(x, f(x))$.

Physically, there are many different ways to morph one shape into another. Mathematically, the mapping of a domain onto a range can have many different *parameterizations*. The only difference is that they trace their range at different speeds (i.e., their derivatives may be different) or by different methods.

Referring to Figure 1 below, x is replaced by $2x$ in the second example, indicating a traversing of domain by twice the original speed, and it shows: we only need half the domain of the first example to do the job. In the third example, it takes the mapping the

[5] Rigorously speaking, the set of integers cannot be proven to exist (without assuming the axiom of infinity, to be mentioned later). However, if such a set satisfying the properties of the integers does exist, it can be proven to be unique. From this logical conundrum stems the famous quote of Kronecker, "God created the integers; all else is the work of man."

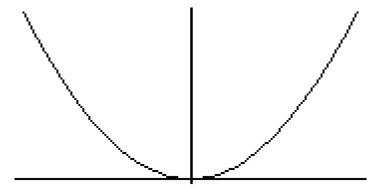
Set	Domain	Range
$\{x, x^2\}$	$[-1, 1]$	
$\{2x, 4x^2\}$	$[-1/2, 1/2]$	
$\{e^x - 1, e^{2x} - 2e^x + 1\}$	$(-\infty, \ln(2)]$	
$\{\sin(x), 1 - \cos^2(x)\}$	$(-\infty, \infty)$	
$\{x, \int_0^x 2t\, dt\}$ — $\{x$, area under $2t$ fom 0 to $x\}$	$[-1, 1]$ (Note: this is the range of x. t is a "dummy variable")	

Table 1

entire domain of $(-\infty, 0)$ just to trace the left half of the range. The right half is traced more quickly; its domain is only $[0, \ln(2)]$. In the third example, the range is actually traced and retraced an infinite number of times, as a ball would roll frictionlessly in a valley. The last example is particularly interesting. Usually we parameterize a point by some geometric concept such as its *distance* from orthogonal axes (Cartesian coordinates) or maybe by a radial *vector* and an *angle* (polar coordinates), but there exist other valid geometrical concepts such as *area*, *inner product*, *cross product*, etc. for which to refer. In this case, the second coordinate is related to the area under the line $2x$. These means to identify points are equally valid, and different methods are at times more useful. This concept is at the core of the usual change of variables method of calculus.

Abstraction

may came home with a smooth round stone
as small as a world and as large as alone.
E. E. Cummings

Mathematics is the art of giving the same name to different things.[6]
Henri Poincaré

If I were in any position to say what the most basic questions math strives to understand, they would be:

- How do things change?
- When do we consider things different, and when do we consider them the same?

These lie at the heart of much mathematics. It is interesting how concrete and seemingly reasonable these questions are, yet how the mathematics springing from them grows increasingly abstract and its connection to practicality more opaque. The mapping-as-a-verb "nuance" is the foundation of the first question, for mappings are the keys to mutations of shape. You can consider a shape as a structure imbedded in space, or even the morphing of space itself to create the structure. This is the heart of the distinction between seeing a parabola imbedded in the plane, or thinking only of the rubber band as being a line, morphed or curved, with no relation to any larger space.

The second question seems too simple-minded: shouldn't two objects be different

[6] As opposed to the quotation: Poetry is the art of giving different names to the same thing.

if there exists some distinguishing element between them? Concisely: no. Difference is defined by assumptions. We first need to set down principles in order to distinguish objects by them. Indeed, without any principles everything is identical to everything else, or equivalently, nothing is identical; there are no distinctions. For instance, are you being pulled down by gravity, or is the floor pushing you up? These are not equal interpretations: general relativity suggests the latter to be more accurate. We often perceive differences through vice or prejudice. A quest of science is to eliminate all apparent diversity to a handful—to reduce your millions of biases to a few, or maybe, some believe, to none, making this world of differences the greatest bias of them all.

Our faith in recognizing likeness and difference is possibly our most naïve, and much of this essay is concerned with the second question. Algebra and Topology have their origins in this question. Algebra focuses on the structure of systems and the relations between their parts. For instance, the positive real numbers equipped with multiplication, written ($\mathbf{R}^+$, $\times$), is no different in structure from the real numbers equipped with addition ($\mathbf{R}$, $+$). Any operation performed in one can be likened to a similar one performed in the other. Concretely, there exists a mapping,[7] $f(x) = e^x$, sending one to the other, and $f^{-1}(x) = \log_e(x)$, reversing the process. For example, 5×6 can be realized by first applying f:

$$f(5) \times f(6) = e^5 \times e^6 = e^{5+6} = f(5 + 6).$$

The operation on the left is multiplication, while on the right we are led to addition, unifying two seemingly different mathematical operations and spaces. This connection, an isomorphism, was exploited by John Napier (1550 - 1617), inventor of logarithms, to make calculations involving multiplication simpler by reducing them to the addition of their logs: a technique needed to handle the gargantuan numbers coming from astronomical data.

This has profound meaning. There is no experiment you can perform, no operation you can conduct to distinguish these two structures. Suppose we are trying to investigate a physical process that multiplies two positive quantities. We might mathematically describe the law governing this process as multiplication, but Nature could in fact be adding logarithms and exponentiating, all to our passive gullibility. However, the question, "what is really happening?" is moot, for they are the same process spoken in different languages. Neither is more valid that the other; both are real. Put in another context, if your blue were my red and vice versa, which of our realities would be more real? Truth is not what you experience, but what motivates experience.

Topology, on the other hand, imposes a radical equivalence on space. For the topologist, two shapes are identical if one can be morphed into the other and back continuously (without jumps or tears). For instance there is no difference between a clenched fist and an open hand because there is a naturally continuous mapping between them: the relaxing and contracting of your fingers. Even stronger, a topologist can't recognize your fingers, since your fingers can be shrunk down into your palm and back continuously! What isn't continuous then? Well, I cannot detach a finger without tearing your skin. Similarly I cannot unwrap a circle into a line, or unwrap a donut into a rod,

[7] Note the similarity between this mapping and $f(x) = x^2$ discussed earlier. Both map $\mathbf{R} \rightarrow \mathbf{R}^+$, but this mapping also switches the operations (multiplication and addition).

without tearing the original spaces somehow. A circle missing a single point, however, can be unwrapped since it is "pre-torn." Similarly a basketball (with a hole for the air pump) can be continuously deformed into a plane (spherical projection). These shapes, though apparently different, are topologically indistinguishable. This may seem as just mathematical abstraction, but they have far-reaching consequences. When we consider the notion of curvature of a surface, it turns out to be a topologically invariant property (specifically, it is invariant under isometries[8]). That means a basketball can be considered (i.e., consistently defined mathematically) as flat as the court you dribble it on, while a sphere (the surface of a ball), by just including that lone point, cannot.

Did I just say what I think I said? Put in a more bizarre context: if you (being two-dimensional) lived in a sphere with a single point missing, and had no ability to reach into any larger space in which the sphere is imbedded, you would not be able to tell whether you lived on a plane or on a "curved" surface, in fact any experiment you performed would confirm your world "flat."[9] When it comes to our universe as a whole, there is no larger space in which spacetime is imbedded; all of our calculations take place within it. Reality may come in many shapes and sizes, all isometric to one another.

Our intuitive geometric notions of distance, curvature, area, etc. when deeply scrutinized become very foreign concepts. Mathematics is exceptionally equipped to abstract the fundamentals of a concept. It has no regard for prejudices and assumptions as we do. With abstractness comes great surprise, beauty and unification. We realize what is really at the heart of things we take for granted and just how little we need to do so much.

Physics and Reality

It is the generation by models of a real without origin or reality: a hyperreal. The territory no longer precedes the map, nor survives it. Henceforth, it is the map that precedes the territory—precession of simulacra.

Simulacra and Simulation – Jean Baudrillard

When we look upon the world we see and feel its multitude of forms: its trees, buildings, people, colors, sensations, etc. We experience chaos, with hardly a semblance of order. As physical bodies and processes surround us, we naively include them in our fundamental description of the world. But is such a suspicion accurate? It is the goal of science, whose method is mathematics, to sift through the glorious charade and simulacra and discover the silent language through which all this difference materializes. Consciousness becomes a terminal through which a very simple external reality is transformed—mapped, if you will—into the brilliantly insane internal reality we experience. In our daily lives we are always ignorant of any other reality beside the one we are incessantly bombarded with. But if isomorphic realities are equally true, there

[8] Isometries can be considered the geometrical equivalent of isomorphisms. Isomorphisms equate two seemingly different algebraic structures, while isometries equate two seemingly different shapes. Two shapes are isometric if there exists a differentiable mapping from one to the other and back preserving distances and angles between tangent vectors (i.e., the inner product, or equivalently, the metric).

[9] Keep in mind that you cannot, for instance, see a distant ship's mast appear before its hull because all your senses, including sight, only operate on the surface and do not extend off it into an imbedding three-dimensional space, but the conclusion is still striking.

may be many ways to view, appreciate, and understand our universe. Physics has made much progress in piercing through the mist pervading our senses to realize that trees, buildings, people, etc. are not fundamental to our world, but are derivatives of more basic forms. Indeed, notions of color, hotness or coldness, hard or soft, fragrant or foul, quiet or loud are not what they seem: colors are electromagnetic waves of various frequencies; temperature is the average velocity of particles; tenacity is the arrangement of molecules and bonds; odor is an interaction between air-born chemicals and nasal receptors; volume is the amplitude of sound waves, etc.

There has never been a moment in which you have actually touched an object; you have only felt the repulsive force of its electrons upon your own. There has never been a time in which you have seen an object; you have only interpreted the waves it emits, waves that are not part of it. All of our senses are fictions that at best refer to some underlying truth. Like numbers and equations, our senses provide an isomorphism between truth and subject—between external and internal reality.

However, this correspondence has been finely tuned by evolution, which may or may not be a trustworthy arbiter. Our senses are instruments of survival, born and developed to prolong existence, not to appreciate truth. There is no mandate on evolution to respond to fact. If believing in a fiction increased our chances of survival, evolution would obliviously steer us in such a direction. These aren't just science fictional musings, this is our reality: the reality our experiences construct. It is why we believe in trees and people, colors and feelings, instead of (and I say this non-absolutely) photons and quarks, waves and fields, or maybe tiny vibrating strings. Just what is out there and what can we disregard as evolutionary trickery?

There are other complications, too. Our experiences rarely deal with the very fast, or the very small, terrains in which our understanding has been revolutionized in the last hundred years by intensive scientific scrutiny. It was believed at the turn of the 19th century that science was rapidly approaching an understanding of all aspects of reality; all that was left were the details (specifically blackbody radiation and the luminiferous aether). But Nature hides her secrets in the details, and exploration of them led to the greatest upheavals in scientific history, quantum mechanics and relativity. It was as if science was gazing upon the world through a pinhole, exposing an area that had been exhausted by research, but leaving huge regions of reality untouched. It is a realizable worry that perhaps there are regions unfamiliar to our experience that we may never explore, and worse yet, regions which may give a hint towards reality in discord with our own present understandings—regions that would reveal a much grander universe, but regions we will never know. The two-fold hope is that with whatever Nature touches, Her residue is left, and no part of Nature can exist independently. And so it is on our shoulders a matter of dedication: if man has the will and capacity to grasp the fiction he has been born into, then he can only be mislead a finite number of times, before being at peace with the universe forever.

Mathematics and Truth

Pure mathematics consists entirely of such assertions as that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true…. Thus

mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.
Betrand Russell

With physics defined as the study of the correspondence between external and internal reality, where does that leave mathematics? Physics must conform to reality; mathematics, however, need only conform to consistency. Its strength is curbed only by human imagination. But like evolution, mathematics cares not to distinguish truth from fiction. Mathematics will readily embrace a fiction if it obeys consistency. This was a revelation that dawned upon the field at the arrival of non-Euclidean geometry.

Standard or Euclidean geometry was based on four very basic postulates and a fifth more mysterious one:

1. Any two points can be connected by a straight line.
2. A finite line may be extended indefinitely in a straight line.
3. A circle may be drawn with any given center and any given radius.
4. All right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Amazingly enough, from these five postulates, all of Euclidean geometry can be deduced, and much was done so in the 13 collected books of Euclid (written around 300 BC). For centuries, however, mathematicians including Euclid himself believed the fifth postulate (dubbed the parallel postulate since it can be used to prove properties of parallel lines), with its nagging complexity, could be deduced from the preceding four. Over the ages many would announce a derivation of the parallel postulate that would later be disproved. Rigorous work ended fruitlessly. Then another approach was taken: Girolamo Saccheri (1667-1733) decided to deny the fifth postulate and see where subsequent deductions took him. He was led to outrageous results, for instance, that the sum of angles of a triangle could be less than 180 degrees. He felt the absurdity of these proofs provided enough convincing evidence that the parallel postulate be true. Prejudice at work. As hard as he and others tried, however, no logical contradictions were ever found. Non-Euclidean geometry was self-consistent.

What then are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true and if the old weights and measures are false, if Cartesian coordinates are true and polar coordinates are false. One geometry cannot be more true than another; it can only be more convenient.
Henri Poincaré

In the 1800s, when Gauss, Lobachevsky, Bolyai and others rigorously developed elliptic and hyperbolic geometry, the foundations of mathematics were shaken. This marked a time when mathematics grew its wings, and flew off the landscape of reality into a far more expansive plane. Before there was a confidence that Euclidean geometry was absolute, and in a sense, real, but now the validity of one geometry over another could only be tested experimentally. They were all "true" in the sense that they were all logically deduced and could be applied to certain circumstances. No one was any more insightful than another, but each provided very rich and exotic structures that could be analyzed rigorously. With Einstein's general relativity (1915), a vastly new universe was revealed exposing Euclidean geometry for what it was: a possibility.

Foundations of Mathematics

From this proposition it will follow, when arithmetical addition has been defined, that 1+1 = 2.

Principia Mathematica (pg. 362) – Russell and Whitehead

Just as there was much concern for geometry's postulates, so too was there for the foundations of mathematics. Almost all of the mathematics you've learned, believe it or not, is derived from nine axioms of logic called the Zermelo-Fraenkel axioms. In plain language they are:

- Axiom of extensionality: Two sets are the same if and only if they have the same elements
- Axiom of the empty set: There exists a set with no elements: the empty set, {}.
- Axiom of paring: If x, y are sets, then there exists a set containing x and y as its only elements.
- Axiom of union: For any set x, there is a set y such that the elements of y are precisely the elements of the elements of x.
- Axiom of infinity: There exists a set x such that {} is in x and whenever y is in x, so is $y \cap \{y\}$.
- Axiom of power set: Every set has power set. That is, for any set x there exists a set y, such that the elements of y are precisely the subsets of x.
- Axiom of regularity: Every non-empty set x contains some element y such that x and y are disjoint sets
- Axiom of separation: Given any set and any proposition P(x), there is a subset of the original set containing precisely those elements x for which P(x) holds.
- Axiom of replacement: Given any set and any mapping, formally defined as a proposition P(x, y) where P(x, y_1) and P(x, y_2) implies $y_1 = y_2$, there is a set containing precisely the images of the original set's elements.[10]

From these axioms the set of natural numbers can be constructed, where we associate (isomorphically speaking):

$$0 = \{\},\ 1 = \{\ \{\}\ \},\ 2 = \{\ \{\},\ \{\ \{\}\ \}\ \},\ldots$$

From here the integers, rational, complex numbers, etc. can be defined, along with operations between them. Then open sets can be defined leading to the definition of limits, continuity, derivatives, functions, etc. etc. etc. It's quite amazing, actually. Everything comes down to these nine axioms. Well, almost everything…

Mathematics through the Looking Glass

Unlike the sciences, mathematics has the ability to turn its analytical lens upon itself: it can be introspective. When mathematicians tampered with the axioms of

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[10] Taken from http://en.wikipedia.org/wiki/Zermelo-Fraenkel_set_theory

geometry very new and striking ideas surfaced. What can we say about the axioms that the very essence of mathematics—logic—rest upon? What about the nine Zermelo-Fraenkel axioms? Are we missing any? Are they all necessary? Do any of they arrive at contradictions? Ten is such a nice number…

It turns out that we cannot know whether the Zermelo-Fraenkel axioms are self-consistent; we take it on faith. Since they form the basis of ordinary mathematics, it is impossible to use ordinary mathematics to prove them (Gödel's second incompleteness theorem). However, it is encouraging that no one has yet found any contradictions. That's strange enough, but there is more mysteriousness. Are there enough of them? Can everything true be proved by them, and everything false disproved by them?

In the early 1900s, grand attempts to axiomatize all of mathematics were proposed, literally ensuring that every true theorem can be indisputably traced back to founding axioms, and any false theorem to a negation of one or more axioms. The most notable attempt may be Russell and Whitehead's three-volume work *Principia Mathematica*, quoted above. Much time was spent deciding upon the logical underpinnings of mathematics, and proving their consistency and completeness, that is, until Kurt Gödel (1906 - 1978) single-handedly debunked the effort. He showed that any consistent axiomatic system powerful enough to construct the set of natural numbers, {0, 1, 2, …} (which assumes a recursive mechanism e.g., the axiom of infinity), will have true statements that cannot be proved or disproved within the system. Hence, like a Heisenberg uncertainty condition, no system can be both complete and consistent.

Gödel was able to prove this by formally coding such statements as, "This statement cannot be proven within this axiomatic system" within the system: a true, yet necessarily unprovable, statement. This is an amazing result because it is a feature of any and all powerful enough formal systems, irrespective of the axioms. You may wish to add axioms to your system in order to prove undecidable statements, but this will only make your larger formal system suffer an equal fate.

However, beyond these introspective paradoxes, the nine Zermelo-Fraenkel axioms fall short in other interesting ways. There are several statements that are independent of the axioms. If the Zermelo-Fraenkel axioms are the whole story, such statements can be considered both true and false; they are undecideable propositions of logic. The most famous of these is the continuum hypothesis, which asserts that there does not exist a set whose cardinality (number of elements) lies between that of the set of integers, {…, -2, -1, 0, 1, 2, …}, and the set of real numbers (e.g., 5, ½, $\pi, \sqrt{2}$, *e*, …). Note that both these sets are infinite in size, however, Georg Cantor (1845 - 1918) showed there exists a hierarchical ranking of infinities. By identifying sets via one-to-one correspondence between their elements, Cantor proved such propositions as the cardinality of the set of rationals (numbers expressible as fractions) is equal to that of the set of integers. This should seem counterintuitive since the integers are contained in the rationals. The size of these sets is called countably infinite (since they can be put into an ordered list and counted) and is denoted by the cardinal number $\aleph_0$ ("aleph-null"). However, no such correspondence exists between the set of reals and the integers; there are just too many real numbers, uncountably many (try constructing an ordered list of them), so the set of reals is given a larger cardinality, $\aleph_1$. The continuum hypothesis can then be restated as: there exists no set whose cardinality lies strictly between $\aleph_0$ and $\aleph_1$.

Maybe the Zermelo-Fraenkel axioms are missing something. In fact, there is a commonly assumed tenth axiom, independent of the others, called the axiom of choice. Simply put, this axiom asserts that if X is a collection of non-empty sets, one can "choose" a member from a set in X. This may seem ridiculous, but for certain collections of sets, a well defined method of choosing elements may not be obvious, or even exist at all if one were to deny this axiom. For instance, if X were the collection of all non-empty sets of real numbers, how would one define a choice over this uncountably infinite collection where many of the sets have no distinguishable element?[11] Bizarre properties are equivalent to the axiom of choice and its negation. If the axiom of choice is indeed true, then it is possible to "carve up" a solid figure of given volume into finitely many pieces (five in the case of a sphere) and, by using only rotation and translation, reassemble the pieces into two solids with the *same* volume as the original (the Banach-Tarski paradox). The pieces themselves are ridiculously complicated, and constructible only in theory, but the insanity of the result remains. On the other hand, the axiom of choice is equivalent to the statement that for any two sets X and Y, either the cardinality of X is less than or equal to Y, or is greater than Y. Hence, the negation of the axiom of choice results in a similarly bizarre proposition: there exist two sets of incomparable size.

Within the philosophy of mathematics are two conflicting paths, Platonism and Formalism. The former believes mathematical entities exist independent of the human mind. Supporters declare the existence of a separate abstract and immutable world in which absolute mathematical truths reside. Mathematicians, then, are not creators of mathematics, but discoverers. In this perspective, the continuum hypothesis and axiom of choice (among other independent propositions) are decidable. The question then becomes, how do we access such a world? Is it by a proper choice of axioms or something else entirely? If so, what axioms are needed to connect our mathematics, to the Platonic ideal? Formalists, on the other hand, deny the existence of such a world, leaving verifiability resting entirely on the shoulders of proof. To them, truth is subjective.

Math, Physics and Reality

The choice, however, of whether to be fully seduced by the face nature reveals directly to our senses, or to also recognize the reality that exists beyond perception, is ours.
Brian Greene

No one really knows why mathematics is so useful at describing physical phenomena. We've been spoiled by its applicability, and so take its success on faith, somehow assuming that mathematics and reality are closely connected. Though mathematics, free to roam, can create any number of possible (i.e., consistent, or logical) worlds, it is physics that deals with the mathematics of our world. In my view, physics is just mathematics until someone says, "and this represents…" (relating the abstract math world to the physical world), or imposes some intuitive symmetry argument (which can be viewed as an additional, physical, axiom).

However, the impurities in the physical representation of our world (from

[11] E.g., the set of all real numbers greater than 0. There is no least element, no greatest element, seemingly no distinguishing feature between elements for which to base a choice.

evolution), and our unreliable intuitive physical beliefs (from prejudice) are barriers to our understanding. In order to really grasp our reality, one needs to shed his potentially corrupt sensory data and any preconceived notions no matter how obvious seeming. All our experiences and observations are results of the interaction between our minds and bodies with an underlying reality, but if there were a way to bypass that interaction and reach the underlying reality directly only then would we grasp the essence of what is out there. We must literally study reality from the perspective of a rational inanimate object—a rock with reason—ignoring our internal reality and focusing only on the external. This requires a physics stripped of the physical. It requires mathematics.

External reality is the same whether you are blind or deaf, man or machine, alive or dead; only internal reality is concerned with such things because it relies on mechanisms of interpretation—it relies on senses. Difference is defined by assumption.

It is interesting to take this a bit further and focus on our most refined sense, sight. Just where exactly does the image of our reality appear? Unlike when we gaze upon a monitor or screen, we can precisely spot a place where an image is produced,[12] there is no such surface onto which reality is projected on. Reality is experienced, and constantly created, internally within the mind. There are no pixels. Our image of the world is the product of chemical disturbances in our brains. A spatially extended reality need not even exist! So it is extremely fascinating: to wonder where the image of our reality appears, when we casually brush it off, assuming it to be "out there" when it is within us all along.

The closest I come to ridding my thoughts of a direct spacetime association is when I'm immersed in a mathematical calculation (often having to do with spacetime!)
The Fabric of the Cosmos – Brian Greene

Many physicists today believe that the most taken-for-granted physical notions, space and time, are illusory. They believe space and time to be derivative concepts of more fundamental mechanisms. This simple idea is most baffling. Personally, it is impossible for me to think of any physical process without framing it in either space or time. But special relativity shows us time is a subjective experience, and quantum mechanics shows that space, on extremely small scales, turns into an ungovernable tumultuous landscape due to the uncertainty principle. This world we have come to know may, in reality, be much stranger than any of us can conceive.

Reality as Epiphenomenon

The point…is simply that meaning can exist on two or more different levels of a symbol-handling system…. The presence of meaning…is determined by whether or not reality is mirrored in an isomorphic (or looser) fashion…whether one's top level is engaged in proving kōans of Boolean Buddhism, or in meditating on theorems of Zen Algebra.
Gödel, Escher, Bach: An Eternal Golden Braid – Douglas Hofstadter

Like geometry's relationship with truth, reality's is similar. There are many ways

[12] Actually, the image is never on the monitor or screen, but rather is emitted from it, further distancing senses from subjects.

to experience the world. The most obvious is through the senses, where the world evolves through color and sound. Then there is through science—the mind's eye—the token of a rational being, which illuminates order from chaos. What is to say there is no way entirely through math? We have learned that truth is a relative term: isomorphic structures are indistinguishable; isometric spaces are identical to those living within them; consistent geometries are on equal footing. Are isomorphic realities likewise valid? Our minds and bodies have grown to interpret reality in a certain, non-unique, way. It may be that there exists a most basic, most fundamental reality, of which scientific insights and natural experiences are translations. If such a reality existed, I would imagine it mathematical in nature. There would be no greater spiritual realization for me than to unify this beautiful world through the power of mathematics. Everything physical would disappear as mere fabrications of a deeper reality, independent of space and time. All we have come to know and experience would emerge as an epiphenomenon—a consequence—of pure math. But I could easily be getting ahead of myself. And the axioms on which reality is built may be physically inspired.

Math, Beauty and Creativity

What the rules of the perfection of divine conduct consist in, and that the simplicity of the ways is in balance with the richness of effects.
Gottfried Leibniz

What is Quality?
Zen and the Art of Motorcycle Maintenance – Robert Pirsig

Quality, beauty, creativity may be impossible to define precisely, but many will agree they depend on using simple devices to create extraordinary effects. In an attempt to reconcile man's abuses of freewill with an omni-benevolent God, the philosopher and mathematician Gottfried Leibnitz justified our world as being "the best of all possible worlds" by comparing the simplicity of means to the richness of their physical consequences. This world is based on very few principles, yet we interact with it and appreciate it on innumerable levels. As people we are capable of the most intense feelings and emotions, broadly spanning depression, elation and all degrees in between. We are capable of surmounting impossible feats, while still prone to careless errors. There is no bound to our will, no lid on our imagination, no threshold to our extent. To us, the world we perceive is a richly complicated, animate, unrelenting mystery…all three families of four fundamental particles of it.

Many people view mathematics and art as opposite sides of a spectrum, and this is not surprising given math's unflattering presentation. But when one leaves the realm of redundant calculations and neatly boxed formulae, when one explores the space behind numbers and equations, one realizes that mathematics is a vast world with more room unexplored than investigated in any textbook, classroom or lab. Thousands of pages of novel mathematics are written each day. They hinge on creativity and imagination. Nobody can accurately define mathematics because it is just that weird and open-ended. It strips away everything unnecessary—the spirit of minimalist thought—and creates in the raw, with no distractive, irrelevant ostentation. You might think that from such stale reservation would result a boring landscape of obviousness, but somehow this isn't so. From analyzing a handful of axioms by pure reason alone an exotic world forms. Like reality, striking sceneries emerge from remarkably humble beginnings. Non-obvious truths are derived from evident axioms. Unexpected connections arise from distant corners. How is it that by accepting nine axioms on faith, we can develop all of number theory and analysis? How is it that logs, originally created to simplify multiplication of large numbers, relate to the integral of the hyperbola? How are

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!} = 2.71828\ldots$$

$$\pi = \text{the ratio of a circle's circumference to its diameter} = 3.14159\ldots$$

$$i = \sqrt{-1}$$

related by $e^{i\pi} + 1 = 0$, uniting analysis, geometry and complex numbers? Understanding mathematics will always be a work in progress as more connections are cemented, but it will always be an exciting and surprising field.

If creativity is doing so much with so little, if it can be judged by the ratio of effects to means, then art, music and literature have to surrender to mathematics as their superior. In this context, it should be no wonder why mathematics is so apt to explore Nature, for She has no patience for pretension and flattery. Natural wonders of the world, from rainforests to canyons, sunsets to rainbows, the depths of oceans to stars in the sky, to life, are not here because Nature decided to spice things up, but because they have to be. They are astonishing results of a much more modest external reality. Like colony behavior from individual ants, economic laws from currency, consciousness from a neural network, and maybe reality from mathematics, these phenomena are not fundamental, but are properties of emergence.

If I had a choice of which reality I would rather experience, between a mathematical one, a scientific one, or the one we perceive, with absolute resolve I would choose the latter. All the mathematical and physical insight we may have will never detract from the beauty of our world; it will only make the illusion more magnificent. But the big picture descends by understanding the correspondences between these realities and from where they stem. This picture stares beyond our perceptions and into the heart of what it means to exist, to be real. Mathematics may be the lens through which truth is exposed, or it may not be. But regardless, in my estimation, this undertaking is the most fulfilling journey within the capabilities of man. We have been given the gift of reason so that from this rock we can fathom the furthest corners of our universe, and the deepest layers of reality. Now it is on our shoulders a matter of dedication.

We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.
T.S. Eliot